Approximations as generalized derivatives: calculus and optimality conditions in multivalued optimization

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Abstract. Using the definition of first and second-order approximations as generalized derivatives of set-valued mappings introduced in [22] we establish both necessary conditions and sufficient conditions for various kinds of solutions to a multivalued vector optimization problem with general inequality constraints. Our results are more applicable than several recent existing ones in many situations as illustrated by examples provided in the paper. We also develop some elements of calculus for approximations

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1. Introduction

Applying generalized derivatives to investigate optimality conditions in nonsmooth optimization has occupied an important part in optimization study with world-wide enormous efforts for almost half a century. There have been various notions of generalized derivatives with different requirements for the existence and effective use. Each of these derivatives is suitably employed for some classes of problems but none is universal. In [1, 14] a generalized derivative of a type of approximations was introduced. Recently in [17, 18, 21] these approximations were used as derivatives to establish both necessary conditions and sufficient ones for various kinds of solutions in vector optimization. Paper [22] was devoted to extend this notion of derivatives to the case of set-valued mappings and set-valued optimization. One of the salient advantages of the mentioned approximations is that even at a point of infinite discontinuity, a mapping may have first and second-order approximations. Furthermore, the definition of approximations includes as special cases many other notions of generalized derivatives (see e.g. Propositions 2.1-2.4 of [17]). As a result optimality conditions using this kind of derivatives often require very relaxed assumptions and contain many results using other kinds of derivatives. Our aim in this paper is to develop calculus rules for approximations as derivatives for set-valued mappings and establish new optimality conditions of orders 1 and 2 for weak, firm and several kinds of proper efficiency in nonsmooth set-valued vector optimization. We supply many examples to indicate advantages of our results over existing ones in the recent literature.

The organization of the paper is as follows. In the remaining part of this section we collect definitions and notations for our later use. Section 2 is devoted to calculus of approximations. Necessary optimality conditions of orders 1 and 2 are discussed in the next Section 3 for weak efficiency of a set-valued vector optimization problem with general inequality constraints. Then these conditions are of course also necessary for other (stronger) efficiencies. In the last Section 4 we establish both first and second order sufficient conditions for firm efficiency, Henig-proper, strong Henig-proper and Benson-proper efficiencies.

Throughout the paper, if not otherwise stated, let X, Y and Z be real normed spaces. Each norm is denoted by $\|.\|$ with no fear of confusions, since the context always shows clearly what space is concerned. Y^* is the topological dual space of Y and $< .,. >$ is the canonical pair between Y and Y^*. B_Y denotes the open unit ball of Y and $B_Y(y, r)$ the open ball centered at y and of radius r. For $S \subseteq X$, int$S$ and cl$S$ stand for its interior and closure, respectively. The cone generated by S is cone$S := \{\lambda x : \lambda \geq 0, x \in S\}$. Let, in this paper, $C \subseteq Y$ and $D \subseteq Z$ be closed convex cones with nonempty interior (we do not assume their pointedness; C being pointed means $C \cap -C = \{0\}$). The (positive) polar cone of C and its quasi-interior are

$$C^* = \{\varphi \in Y^* :< \varphi, c >\geq 0, \forall c \in C\},$$

$$C^{*i} = \{\varphi \in C^* :< \varphi, c >> 0, \forall c \in C \setminus \{0\}\}.$$

For the cone D, $D(z_0) :=$ cone$(D + z_0)$. A nonempty subset B of a convex cone C is called a base of C if $C =$ cone$B$ and $0 \notin$ cl$B$. For positive t, the notation $o(t)$ is used for a moving point satisfying $t^{-1}\|o(t)\| \to 0^+$ as $t \to 0^+$. For a multivalued (i.e. set-valued) mapping $F : X \to 2^Y$, the domain, graph and epigraph of F are

$$\text{dom}F = \{x \in X : F(x) \neq \emptyset\},$$

$$\text{gr}F = \{(x, y) \in X \times Y : y \in F(x)\},$$

$$\text{epi}F = \{(x, y) \in X \times Y : y \in F(x) + C\}.$$

We write $x \xrightarrow{F} x_0$ if $x \in$ dom$F$ and $x \to x_0$. F is said to be calm at $(x_0, y_0) \in$ gr$F$ if there exist $L > 0$ and a neighborhood U of x_0 such that, for all $x \in U \cap$ dom$F$,

$$F(x) \subseteq y_0 + L \parallel x - x_0 \parallel \text{cl}B_Y.$$

For single-valued mappings calmness is relatively weak (weaker than local Lipschitz) but for set-valued maps it is quite strong. We use also the following more relaxed concept. F is called C-calm at (x_0, y_0) if there exist $L > 0$ and a neighborhood U of x_0 satisfying, for all $x \in U \cap$dom$F$,

$$F(x) \subseteq y_0 + C + L \parallel x - x_0 \parallel \text{cl}B_Y.$$

Clearly, F is C-calm at (x_0, y_0) if and only if $F(.) + C$ is calm at (x_0, y_0). A subset S of X is called star-shaped at x_0 if, for all $x \in S$, all $\alpha \in [0, 1]$, $(1-\alpha)x_0 + \alpha x \in S$. F is said to be C-star-shaped at x_0 on S, where S is star-shaped at x_0, if for all $x \in S$ and all $\alpha \in [0, 1]$, one has

$$(1-\alpha)F(x_0) + \alpha F(x) \subseteq F((1-\alpha)x_0 + \alpha x) + C.$$

If S is convex and this relation is fulfilled for all x and x_0 in S then F is called C-convex. F is said to be pseudoconvex at $(x_0, y_0) \in \mathrm{gr}F$ if

$$\mathrm{epi}F \subseteq (x_0, y_0) + T_{\mathrm{epi}F}(x_0, y_0),$$

where, for $A \subseteq X$,

$$T(A, x_0) := \{u \in X : \exists t_n \to 0^+, \exists u_n \to u, \forall n, x_0 + t_n u_n \in A\}.$$

is the contingent (or Bouligand) cone of A at x_0 [4].

In [5] the arcwise-connectedness was introduced as follows. A subset S of X is called arcwise-connected at x_0 if for each $x \in S$, there exists a continuous arc $H_{x_0,x}(t)$ defined on $[0, 1]$ such that $H_{x_0,x}(0) = x_0, H_{x_0,x}(1) = x$ and $H_{x_0,x}(\alpha) \in S$ for all $\alpha \in (0, 1)$. F is said to be C-arcwise connected at x_0 on S, where S is arcwise-connected at x_0, if for all $x \in S$ and all $\alpha \in [0, 1]$,

$$(1-\alpha)F(x_0) + \alpha F(x) \subseteq F(H_{x_0,x}(\alpha)) + C.$$

Remark 1.1. Every convex or star-shaped set is arcwise-connected and every C-star-shaped mapping is C-arcwise-connected. However, the converse is not true as shown by the following example.

Example 1.2. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 \geq x_2\}$ and $F : X \to 2^Y$ be defined by

$$F(x_1, x_2) = \begin{cases} [x_2, +\infty), & \text{if} \quad x_1^2 \geq x_2, \\ (-1, 1), & \text{otherwise.} \end{cases}$$

S is not convex since $x = (0, 0)$ and $y = (1, 1)$ are in S but $\frac{1}{2}x + \frac{1}{2}y = (\frac{1}{2}, \frac{1}{2}) \notin S$. F is not C-convex because, for $x = (-2, 0)$ and $y = (0, 2)$,

$$\frac{1}{2}F(x) + \frac{1}{2}F(y) \subsetneq F(\frac{1}{2}x + \frac{1}{2}y) + C.$$

Furthermore, S is not star-shaped at $x^0 = (0, 0)$ and so F is not C-star-shaped at x_0. However, it is easy to check that S is arcwise-connected at any $x \in S$ and F is C-arcwise connected at any $x \in S$ on S with the continuous arc $H_{x,y}$ defined by $H_{x,y}(t) = (\sqrt{(1 - tx_1^2 + ty_1^2}, (1-t)x_2 + ty_2)$ for $t \in [0, 1]$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Like in [17, 18, 21], for discussing optimality conditions we impose the following relaxed compactness assumption. Let $L(X, Y)$ stand for the space of the continuous linear mappings from X to Y and $B(X, X, Y)$ for that of the continuous bilinear mappings from $X \times X$ to Y.

Definition 1.3. (i) Let M_n and M be in $L(X,Y)$. The sequence $\{M_n\}$ is said to pointwise converge to M and written as $M_n \xrightarrow{p} M$ or $M = \mathrm{p}-\lim M_n$ if $\lim M_n(x) = M(x)$ for all $x \in X$. A similar definition is adopted for $N_n, N \in B(X,X,Y)$.

(ii) A subset A of $L(X,Y)$ (of $B(X,X,Y)$) is said to be (sequentially) asymptotically p-compact if

(a) for each norm bounded sequence $\{M_n\}$ in A, there is a subsequence $\{M_{n_k}\}$ converging pointwise to some $M \in L(X,Y)$ ($M \in B(X,X,Y)$, respectively).

(b) for each norm unbounded sequence $\{M_n\}$ in A, which can be assumed with $\|M_n\| \to \infty$, the sequence $\{M_n/\|M_n\|\}$ has a subsequence which pointwise converges to some $M \in L(X,Y)\setminus\{0\}$ ($M \in B(X,X,Y)\setminus\{0\}$, respectively).

Remark 1.4.

(i) If X and Y are finite dimensional, a convergence occurs if and only if the corresponding pointwise convergence does, but in general the "if" does not hold. Even when $Y = \mathbb{R}$ and hence the pointwise convergence coincides with the star-weak convergence, it differs from convergence if X is infinite dimensional. For general norm spaces X and Y, the pointwise convergence is corresponding to a nonmetrizable topology.

(ii) In this paper we are always concerned only sequential convergence. Hence we omit the term "sequentially" in Definition 1.3(ii).

(iii) Assume that $\{P_n\} \subseteq L(X,Y)$ is norm bounded. If $x_n \to x$ and $P_n \xrightarrow{\mathrm{p}} P$, then $P_n x_n \to Px$. Indeed, it follows directly from the following inequalities

$$\begin{aligned}\|P_n x_n - Px\| &\le \|P_n x_n - P_n x\| + \|P_n x - Px\| \\ &\le \|P_n\|\|x_n - x\| + \|P_n x - Px\|.\end{aligned}$$

A norm bounded sequence $\{N_n\} \subseteq B(X,X,Y)$ possesses a similar property.

For $A \subseteq L(X,Y)$ and $B \subseteq B(X,X,Y)$, we adopt the following notations:

(1) $\mathrm{p}-\mathrm{cl}A = \{M \in L(X,Y)/\exists\{M_n\} \subseteq A,\ M = \mathrm{p}-\lim M_n\}$,
(2) $\mathrm{p}-\mathrm{cl}B = \{N \in B(X,X,Y)/\exists\{N_n\} \subseteq B,\ N = \mathrm{p}-\lim N_n\}$,
(3) $A_\infty = \{M \in L(X,Y)/\exists\{N_n\} \subseteq A,\ \exists t_n \to 0^+,\ M = \lim t_n M_n\}$,
(4) $\mathrm{p}{-}A_\infty = \{M \in L(X,Y)/\exists\{M_n\} \subseteq A,\ \exists t_n \to 0^+,\ M = \mathrm{p}-\lim t_n M_n\}$,
(5) $\mathrm{p}{-}B_\infty = \{N \in B(X,X,Y)/\exists\{N_n\} \subseteq B,\ \exists t_n \to 0^+,\ N = p-\lim t_n N_n\}$,
(6) $\mathrm{p}{-}A = \mathrm{p}-\mathrm{cl}A \cup (p - A_\infty \setminus \{0\})$,
(7) $\mathrm{p}{-}B = \mathrm{p}-\mathrm{cl}B \cup (p - B_\infty \setminus \{0\})$.

The sets (1), (2) are pointwise closures; (3) is just the known definition of the recession cone of a set A (not necessarily convex). So (4), (5) are pointwise recession cones. While (6) is the union of (1) and (4) with 0 removed; (7) is similar.

2. Approximations as generalized derivatives for multifunctions

Definition 2.1. [22]

Let $F : X \to 2^Y$ and $(x_0, y_0) \in \mathrm{gr}F$.

(i) A subset $A_F(x_0, y_0) \subseteq L(X, Y)$ is said to be a (first-order) approximation of F at (x_0, y_0) if there exists a neighborhood U of x_0 such that, for all $x \in U \cap \mathrm{dom}F$, there is $r > 0$ satisfying $\|x - x_0\|^{-1} r \to 0$ as $x \to x_0$ and

$$(F(x) - y_0) \cap (A_F(x_0, y_0)(x - x_0) + rB_Y) \neq \emptyset. \tag{2.1}$$

(ii) A subset $A_F^S(x_0, y_0) \subseteq L(X, Y)$ is called a (first-order) strong approximation of F at (x_0, y_0) if there exists a neighborhood U of x_0 such that instead of (2.1) one has

$$(F(x) - y_0) \subseteq A_F^S(x_0, y_0)(x - x_0) + rB_Y.$$

Definition 2.2. [22]
Let $F : X \to 2^Y$ and $(x_0, y_0) \in \mathrm{gr}F$.

(i) A pair $(A_F(x_0, y_0), B_F(x_0, y_0))$ with $A_F(x_0, y_0) \subseteq L(X, Y)$ and $B_F(x_0, y_0) \subseteq B(X, X, Y)$ is called a second-order approximation of F at (x_0, y_0) if $A_F(x_0, y_0)$ is a first-order approximation of F at (x_0, y_0) and there exists a neighborhood U of x_0 such that $\forall x \in U \cap \mathrm{dom}F$, $\exists r > 0$ such that $\|x - x_0\|^{-2} r \to 0$ as $x \to x_0$ and

$$(F(x) - y_0) \cap (A_F(x_0, y_0)(x - x_0) + B_F(x_0, y_0)(x - x_0, x - x_0) + rB_Y) \neq \emptyset. \tag{2.2}$$

(ii) A pair $(A_F^S(x_0, y_0), B_F^S(x_0, y_0))$ where $A_F^S(x_0, y_0) \subseteq L(X, Y)$ and $B_F^S(x_0, y_0) \subseteq B(X, X, Y)$ is termed a second-order strong approximation of F at (x_0, y_0) if $A_F^S(x_0, y_0)$ is a first-order strong approximation of F at (x_0, y_0) and in the place of (2.2) one has

$$F(x) - y_0 \subseteq A_F^S(x_0, y_0)(x - x_0) + B_F^S(x_0, y_0)(x - x_0, x - x_0) + rB_Y.$$

Remark 2.3. (i) If $F = f$ is single-valued the definitions of approximations and strong approximations coincide and collapse to those in [1, 14] if rB_Y is replaced by $o(\|x - x_0\|)$ for approximations and $o(\|x - x_0\|^2)$ for second-order approximations. Approximations are not unique. In particular, any superset of an approximation is also an approximation. In the sequel, for convenience we simply write $A_F(x_0, y_0)$ and $(A_F(x_0, y_0), B_F(x_0, y_0))$ for an (arbitrary) approximation and second-order approximation, respectively, without mentioning that they are so. A similar convention is taken for strong approximations.

(ii) If a first-order strong approximation of F at (x_0, y_0) is norm bounded then F is calm at (x_0, y_0).

To compare approximations with some generalized derivatives of multivalued mappings we recall needed notions. The Kuratowski-Panlevé upper limit $\mathrm{limsup}_{x \to x_0} F(x)$ is the set of all cluster points of all sequences $y_n \in F(x_n)$ as $x_n \xrightarrow{F} x_0$.

Definition 2.4. Let $F : X \to 2^Y$ and $(x_0, y_0) \in \mathrm{gr}F$.

(i) [4] The contingent derivative of F at (x_0, y_0) is the multivalued mapping $D_C F(x_0, y_0) : X \to 2^Y$ defined by

$$\begin{aligned} D_C(x_0, y_0)(v) &= \mathrm{limsup}_{x \xrightarrow{F} x_0, t \to 0^+} \frac{1}{t}(F(x_0 + tv) - y_0) \\ &= \{u \in Y | \ \exists t_n \to 0^+, \exists (u_n, v_n) \to (u, v), \forall n, y_n := y_0 + t_n u_n \in F(x_0 + t_n v_n)\}. \end{aligned}$$

(ii) [26] The Shi derivative of F at (x_0, y_0) is the multivalued mapping $D_S F(x_0, y_0) : X \to 2^Y$ defined by

$$D_S(x_0, y_0)(v) = \{u \in Y |\ \exists t_n > 0, \exists (u_n, v_n) \to (u, v) : t_n v_n \to 0, \forall n, y_n := y_0 + t_n u_n \\ \in F(x_0 + t_n v_n)\}.$$

(iii) [19, 20] The variational set of F at (x_0, y_0) is

$$V^1 F(x_0, y_0) = \text{limsup}_{x \xrightarrow{F} x_0, t \to 0^+} \frac{1}{t}(F(x) - y_0).$$

(iv) [8] The Dini derivative of F at (x_0, y_0) in the direction v is

$$F'(x_0, y_0, v) = \text{limsup}_{t \to 0^+} \frac{1}{t}(F(x + tv) - y_0).$$

Proposition 2.5. *Assume that* $(x_0, y_0) \in \text{gr}F$ *and, for all* $v \in X$, $0 \notin \text{p}-(A^S_F(x_0, y_0)_\infty \setminus \{0\})(v)$. *Then*

$$F'(x_0, y_0, v) \subseteq D_C F(x_0, y_0)(v) \subseteq D_S F(x_0, y_0)(v) \subseteq \text{p} - \text{cl}A^S_F(x_0, y_0)(v).$$

Proof. Only the last inclusion is not clear and needs to be checked. Let $u \in D_S F(x_0, y_0)(v)$. Then, for sufficiently large n, we have

$$t_n u_n \in F(x_0 + t_n v_n) - y_0 \subseteq t_n A^S_F(x_0, y_0)(v_n) + r_n B_Y.$$

So there exists $P_n \in A^S_F(x_0, y_0)$ and $y_n \in r_n B_Y$ such that

$$u_n = P_n(v_n) + y_n/t_n. \tag{2.3}$$

If $\{P_n\}$ is norm unbounded, by the asymptotic p-compactness of $A^S_F(x_0, y_0)$, we can assume that $P_n/\|P_n\| \xrightarrow{p} P \in \text{p}-A^S_F(x_0, y_0)_\infty \setminus \{0\}$. Dividing (2.3) by $\|P_n\|$ and letting $n \to \infty$ we obtain $0 = P(v)$ contradicting the assumption. In consequence, $\{P_n\}$ must be norm bounded. Again by the asymptotic p-compactness, $P_n \xrightarrow{p} P \in \text{p} - \text{cl}A^S_F(x_0, y_0)$. Letting $n \to \infty$ in (2.3) we arrive at $u = P(v) \in \text{p} - \text{cl}A^S_F(x_0, y_0)(v)$ as required. □

Now, we develop some calculus rules for approximations. The proof of the following proposition is direct and hence omitted.

Proposition 2.6. *Let* $F_i : X \to 2^Y$ *for* $i = 1, 2, \ldots, n$.

(i) *For* $(x_0, y_0) \in \bigcup_{i=1}^n \text{gr}F_i$, $\bigcup_1^n A_{F_i}(x_0, y_0)$ *and* $\bigcup_1^n A^S_{F_i}(x_0, y_0)$ *are an approximation and strong approximation, respectively, of* $\bigcup_1^n F_i$ *at* (x_0, y_0).

(ii) *If* $(x_0, y_0) \in \bigcap_{i=1}^n \text{gr}F_i$, *then* $\bigcap_1^n A^S_{F_i}(x_0, y_0)$ *is a strong approximation of* $\bigcap_1^n F_i$ *at* (x_0, y_0).

(iii) *For* $\lambda_i \in \mathbb{R}$, $A_{F_i}(x_0, y_{0i})$ *and* $A^S_{F_i}(x_0, y_{0i})$ *are an approximation and strong approximation, respectively, of* F_i *at* (x_0, y_{0i}), *for* $i = 1, 2, \ldots, n$, $\sum_1^n \lambda_i A_{F_i}(x_0, y_{0i})$ *and* $\sum_1^n \lambda_i A^S_{F_i}(x_0, y_{0i})$ *are an approximation and strong approximation, respectively, of* $\sum_1^n \lambda_i F_i$ *at* $(x_0, \sum_1^n \lambda_i y_{0i})$.

(iv) *Let* $F : X \to 2^Y$, $G : X \to 2^Z$, $(F \times G)(x) := F(x) \times G(x)$ *for all* $x \in X$ *be the Cartesian product and* $(x_0, y_0, z_0) \in \text{gr}(F, G)$. *Then* $A_F(x_0, y_0) \times A_G(x_0, z_0)$ *and* $A^S_F(x_0, y_0) \times A^S_G(x_0, z_0)$ *are an approximation and strong approximation, respectively, of* $F \times G$ *at* (x_0, y_0, z_0).

The following example shows that for approximations Proposition 2.6(ii) is no longer true.

Example 2.7. Let $X = Y = \mathbb{R}$, $F : X \to 2^Y$, $G : X \to 2^Z$ be defined by $F(x) = \{0, x\}$, $G(x) = \{0, -x\}$ and $x_0 = y_0 = 0$. We can take $A_F(x_0, y_0) = \{1\}$ and $A_G(x_0, y_0) = \{-1\}$. Then $A_F(x_0, y_0) \cap A_G(x_0, y_0) = \emptyset$ is evidently not an approximation of $F \cap G$ at (x_0, y_0). (For the corresponding strong approximations, we get $A^S_F(x_0, y_0) = \{0, 1\}$ and $A^S_G(x_0, y_0) = \{0, -1\}$ and then $A^S_F(x_0, y_0) \cap A^S_G(x_0, y_0) = \{0\}$ is a strong approximation of $F \cap G$ at (x_0, y_0).)

Assume that Y is a Hilbert space. The inner product $\langle F, G\rangle : X \to 2^Y$ of F and G is defined by $\langle F, G\rangle(x) := \bigcup_{u\in F(x), v\in G(x)} \langle u, v\rangle$.

Proposition 2.8. (Inner Product). *Let Y be a Hilbert space. Let $x_0 \in \mathrm{dom}F \cap \mathrm{dom}G$, $y_0 \in \langle F, G\rangle(x_0)$ and $u \in F(x_0), v \in G(x_0)$ be such that $\langle u, v\rangle = y_0$. Assume that F and G are calm at (x_0, u) and (x_0, v), respectively. Then $\langle u, A^S_G(x_0, v)\rangle + \langle v, A^S_F(x_0, u)\rangle$ is a strong approximation of $\langle F, G\rangle$ at (x_0, y_0).*

Proof. For x sufficiently close to x_0, there are positive r_F and r_G such that $\|x - x_0\|^{-1} r \to 0$ for $r = \max\{r_F, r_G\}$ and

$$\langle F, G\rangle(x) - y_0 = \langle u, G(x) - v\rangle + \langle v, F(x) - u\rangle + \langle F(x) - u, G(x) - v\rangle$$

$$\subseteq \langle u, A^S_G(x_0, v)(x - x_0) + r_G B_Z)\rangle + \langle v, A^S_F(x_0, u)(x - x_0) + r_F B_Y)\rangle + \langle F(x) - u, G(x) - v\rangle$$

$$\subseteq \Big(\langle u, A^S_G(x_0, v)\rangle + \langle v, A^S_F(x_0, u)\rangle\Big)(x - x_0) + \langle u, r_G B_Z\rangle + \langle v, r_F B_Y\rangle + \langle F(x) - u, G(x) - v\rangle.$$

Because of the calmness of F and G at (x_0, u) and (x_0, v), respectively, there exist L_1 and L_2 such that, for x sufficiently close to x_0,

$$\langle F(x) - u, G(x) - v\rangle \subseteq L_1 L_2 \|x - x_0\|^2 \langle \mathrm{cl}B_Y, \mathrm{cl}B_Z\rangle \subseteq L_1 L_2 \|x - x_0\|^2.$$

Summarizing the above estimates we get, for some positive $\bar{r}$ of higher order than $\|x - x_0\|$,

$$\langle F, G\rangle(x) - y_0 \subseteq \Big(\langle u, A^S_G(x_0, v)\rangle + \langle v, A^S_F(x_0, u)\rangle\Big)(x - x_0) + \bar{r} B_Y.$$

as desired.

□

Consider $F : X \to 2^Y$ and $G : Y \to 2^Z$, the composition $H := G \circ F$ and the resultant multifunction $M : X \times Z \to 2^Y$ defined by $M(x, z) := F(x) \cap G^{-1}(z)$ (observe that $\mathrm{dom}M = \mathrm{gr}H$).

Proposition 2.9. (Chain Rule). *Let $(x_0, z_0) \in \mathrm{gr}H; = \mathrm{gr}(G \circ F)$ and $y_0 \in M(x_0, z_0)$. If F is calm at (x_0, y_0) then $A^S_G(y_0, z_0) \circ A^S_F(x_0, y_0)$ is a strong approximation of H at (x_0, z_0).*

Proof. Since F is calm at (x_0, y_0), when x is sufficiently close to x_0, all $y \in F(x)$ is close to y_0. By the definition of $A_G^S(y_0, z_0)$ one has

$$G(y) - z_0 \subseteq A_G^S(y_0, z_0)(y - y_0) + r_G B_Z.$$

Using the definition of $A_F^S(x_0, y_0)$, for all y as above,

$$y - y_0 \in A_F^S(x_0, y_0)(x - x_0) + r_F B_Y.$$

Hence

$$G(y) - z_0 \subseteq A_G^S(y_0, z_0)(A_F^S(x_0, y_0)(x - x_0) + r_F B_Y) + r_G B_Z.$$

We claim that $u\|x - x_0\|^{-1} \to 0$, for all $u \in A_G^S(y_0, z_0)(r_F B_Y) + r_G B_Z$. Indeed, for such a u there exists $P \in A_G^S(y_0, z_0))$ such that

$$u\|x - x_0\|^{-1} \in (P(r_F B_Y) + r_G B_Z)\|x - x_0\|^{-1}.$$

Clearly, $P(r_F B_Y)\|x - x_0\|^{-1} \to 0$ as $x \to x_0$. Due to the calmness of F,

$$\lim r_G \|x - x_0\|^{-1} B_Z \leq \lim r_G \|y - y_0\|^{-1} \frac{L\|x - x_0\|}{\|x - x_0\|} = 0$$

and we get the claimed assertion. Now we can take positive $\bar{r}$ such that $\bar{r}\|x - x_0\|^{-1} \to 0$ and

$$G(y) - z_0 \subseteq A_G^S(y_0, z_0) \circ A_F^S(x_0, y_0)(x - x_0) + \bar{r} B_Z.$$

Consequently,

$$(G \circ F)(x) - z_0 = \bigcup_{y \in F(x)} (G(y) - z_0) \subseteq A_G^S(y_0, z_0) \circ A_F^S(x_0, y_0)(x - x_0) + \bar{r} B_Z,$$

which is the desired relation.

□

The following example confirms the essentialness of the calmness assumed in Proposition 2.9.

Example 2.10. Let $X = Y = Z = \mathbb{R}$, $F : X \to 2^Y$ and $G : X \to 2^Z$ be defined by $G(y) = \{y^3\}$,

$$F(x) = \begin{cases} \{y \in \mathbb{R} : y \geq x\}, & \text{if} \quad x \geq 0, \\ \emptyset, & \text{if} \quad x < 0 \end{cases}$$

and $x_0 = y_0 = z_0 = 0$. We can take $A_G^S(y_0, z_0) = \{0\}$ and $A_F^S(x_0, y_0) = [1, +\infty)$, Then $A_G^S(y_0, z_0) \circ A_F^S(x_0, y_0) = \{0\}$ is evidently not a strong approximation of $G \circ F$ at (x_0, z_0). The reason is that F is not calm at (x_0, y_0).

The chain rule is not valid for approximations as seen now.

Example 2.11. Let $X = Y = Z = \mathbb{R}$, $F : X \to 2^Y$ and $G : X \to 2^Z$ be defined by

$$F(x) = \begin{cases} \{y \in \mathbb{R} : 0 \le y \le x\}, & \text{if} \quad x \ge 0, \\ \{y \in \mathbb{R} : -x^2 \le y < 0\}, & \text{if} \quad x < 0, \end{cases}$$

$$G(y) = \begin{cases} \{z \in \mathbb{R} : z \ge 0\}, & \text{if} \quad y \ge 0, \\ \{\frac{1}{y}\}, & \text{if} \quad y < 0 \end{cases}$$

and $x_0 = y_0 = z_0 = 0$. We can take $A_G(y_0, z_0) = \{0\} \cup (\alpha, +\infty)$ (where α is positive and fixed) and $A_F(x_0, y_0) = \{0\}$. One has

$$(G \circ F)(x) = \begin{cases} \{z \in \mathbb{R} : z \ge 0\}, & \text{if} \quad x \ge 0, \\ \{z \in \mathbb{R} : z \le -\frac{1}{x^2}\}, & \text{if} \quad x < 0. \end{cases}$$

Hence, $A_G(y_0, z_0) \circ A_F(x_0, y_0) = \{0\}$ is evidently not an approximation of $G \circ F$ at (x_0, z_0). (For the strong approximations $A_F^S(x_0, y_0) = [0, 1]$ and $A_G^S(x_0, y_0) = [0, +\infty)$, $A_G^S(y_0, z_0) \circ A_F^S(x_0, y_0) = [0, +\infty)$ is in fact a strong approximation of $G \circ F$ at (x_0, z_0).)

3. Necessary optimality conditions

We consider the following constrained multivalued optimization problem

$$(\text{P}) : \qquad \min F(x) \qquad \text{s.t.} \quad G(x) \cap -D \neq \emptyset,$$

where $F : X \to 2^Y$ and $G : X \to 2^Z$. We denote the feasible regions by $A := \{x \in X : G(x) \cap -D \neq \emptyset\}$ and $F(A) = \bigcup_{x \in A} F(x)$. (As before, $C \subseteq Y$ and $D \subseteq Z$ are closed convex cones with nonempty interior.) Let $x_0 \in A$ and $z_0 \in G(x_0) \cap -D$. Dealing with optimality conditions, in the remaining part of the paper we always impose the following relaxed compactness assumption

(CA) The approximations and strong approximations under considerations of F and G are asymptotically p-compact.

Recall now the definitions of several approximate cones. The cone of weak feasible directions to A at x_0 is

$$W_f(A, x_0) := \{u \in X : \exists t_n \to 0^+, \forall n, x_0 + t_n u \in A\}.$$

The contingent (or Bouligand) cone of A at x_0 is

$$T(A, x_0) := \{u \in X : \exists t_n \to 0^+, \exists u_n \to u, \forall n,\ x_0 + t_n u_n \in A\}.$$

Recall that $D(z_0) = \{\lambda(d + z_0) : \lambda \ge 0, d \in D\}$. Note that its positive polar cone is

$$D(z_0)^* = (D + z_0)^* = \{d^* \in D^* : \langle d^*, z_0 \rangle = 0\}.$$

We will be concerned with the following kinds of solutions to problem (P).

Definition 3.1. For problem (P), let $x_0 \in A$ and $y_0 \in F(x_0)$.

(i) A pair (x_0, y_0) is a called a local weak efficient solution of (P) if there exists a neighborhood U of x_0 such that
$$(F(A \cap U) - y_0) \bigcap (-\mathrm{int}C) = \emptyset.$$

(ii) For an integer $m \geq 1$, a pair (x_0, y_0) is said to be a local firm (or strict/isolated) efficient solution of order m if (a) $y_0 \in \mathrm{StrMin}_C F(x_0)$ (i.e. y_0 is a strict (Pareto) efficient point of $F(x_0)$ which means $(F(x_0) - y_0) \cap (-C \setminus \{0\}) = \emptyset$) and (b) there exist a neighborhood U of x_0 and a constant $\alpha > 0$ such that, for all $x \in A \cap U \setminus \{x_0\}$,
$$(F(x) + C) \bigcap B(y_0, \alpha \|x - x_0\|^m) = \emptyset.$$

(iii) A pair (x_0, y_0) is termed a local Henig-properly efficient solution if there exist a neighborhood U of x_0 and a pointed convex cone $H \subseteq Y$ with $C \setminus 0 \subseteq \mathrm{int}H$ such that
$$(F(A \cap U) - y_0) \bigcap (-H) = \{0\}.$$

(iv) When C has a base B, a pair (x_0, y_0) is called a local strong Henig-properly efficient solution if there exist a neighborhood U of x_0 and $\epsilon \in (0, \delta)$ such that
$$(F(A \cap U) - y_0) \bigcap (-\mathrm{int}C_\epsilon(B)) = \emptyset.$$
where $\delta = \inf\{\|b\| : b \in B\}$ and $C_\epsilon(B) = \mathrm{cone}(B + \epsilon B_Y)$.

(v) (x_0, y_0) is called a local Benson-properly efficient solution of (P) if there exists a neighborhood U of x_0 such that
$$\mathrm{clcone}(F(U \cap A) + C - y_0) \cap -C = \{0\}.$$

Note that many notions of properness in vector optimization have been introduced and studied in the literature (since each of them is significant in some respect, but none is universal). Only a few of them are dealt with here. To have a relatively comprehensible presentations and comparisons of these kinds of solutions the reader is referred to e.g. [12, 13, 15, 16, 19, 24]. The following properties of the cone of weak feasible directions are often in use in the sequel.

Proposition 3.2. *For $u \notin W_f(A, x_0)$ the following properties hold.*
(i) *There exists $Q \in A_G(y_0, z_0)$ such that $Q(u) \notin -\mathrm{int}(D + z_0)$.*
(ii) *If $z_0 + A_G(y_0, z_0)(u) \subseteq -D$, then for some $N \in B_G(y_0, z_0)$, $N(u, u) \notin -\mathrm{int}(D + z_0)$.*

Proof. (i) Let $z_0 \in G(x_0) \cap -D$. Suppose to the contrary that
$$z_0 + A_G(x_0, z_0)(u) \in -\mathrm{int}D.$$
By the definition of $A_G(x_0, z_0)$, for large n one has, for some positive r with $rn^{-1} \to 0$,
$$G(x_0 + \frac{1}{n}u) \bigcap \Big(z_0 + \frac{1}{n}A_G(x_0, z_0)(u) + rB_Z\Big) \neq \emptyset.$$
Then
$$G(x_0 + \frac{1}{n}u) \bigcap \Big((1 - \frac{1}{n})z_0 + \frac{1}{n}\Big(z_0 + A_G(x_0, z_0)(u) + nrB_Z\Big)\Big) \neq \emptyset.$$

As $z_0 \in -D$ and $z_0 + A_G(x_0, z_0)(u) + nrB_Z \subseteq -\mathrm{int}D$, one sees further, for large n,

$$(1 - \frac{1}{n})z_0 + \frac{1}{n}\Big(z_0 + A_G(x_0, z_0)(u) + nrB_Z\Big) \subseteq -D.$$

This leads to a contradiction to the assumption that $u \notin W_f(A, x_0)$.

(ii) Suppose that

$$z_0 + B_G(x_0, z_0)(u, u) \in -\mathrm{int}D.$$

As $(A_G(x_0, z_0), B_G(x_0, z_0))$ is a second-order approximation, for sufficiently large n one gets, for some positive r with $rn^{-2} \to 0$,

$$G(x_0 + \frac{1}{n}u) \bigcap \Big(z_0 + \frac{1}{n}A_G(x_0, z_0)(u) + \frac{1}{n^2}B_G(x_0, z_0)(u, u) + rB_Z\Big) \neq \emptyset.$$

In consequence,

$$G(x_0 + \frac{1}{n}u) \bigcap \Big((1 - \frac{1}{n} - \frac{1}{n^2})z_0 + \frac{1}{n}(z_0 + A_G(x_0, z_0)(u)) + \frac{1}{n^2}(z_0 + B_G(x_0, z_0)(u, u) + n^2 rB_Z)\Big) \neq \emptyset.$$

In view of the facts $z_0 \in -D$, $z_0 + A_G(x_0, z_0)(u) \subseteq -D$ and $z_0 + B_G(x_0, z_0)(u, u) + n^2 rB_Z \subseteq -\mathrm{int}D$, one obtains

$$(1 - \frac{1}{n} - \frac{1}{n^2})z_0 + \frac{1}{n}(z_0 + A_G(x_0, z_0)(u)) + \frac{1}{n^2}(z_0 + B_G(x_0, z_0)(u, u) + n^2 rB_Z) \subseteq -D.$$

This is a contradiction since u is not a weak feasible direction. □

The following necessary optimality condition is exploited to get Fritz John optimality conditions in the sequel.

Proposition 3.3. *Let assumption* (CA) *be satisfied. Let* (x_0, y_0) *be a local weak efficient solution of (P),* $z_0 \in G(x_0) \cap -D$ *and* $u \in W_f(A, x_0)$*. Then*

(i) *there exists* $P \in \text{p}{-}A_F(x_0, y_0)$ *such that* $P(u) \notin -\mathrm{int}C$*;*

(ii) *if* $A_F(x_0, y_0)(u) \subseteq -C$*, then there is* $M \in \text{p}{-}B_F(x_0, y_0)$ *with* $M(u, u) \notin -\mathrm{int}C$*.*

Proof. (i) For a sequence $t_n \to 0^+$ and n large enough we have (for a positive r_n such that $r_n t_n^{-1} \to 0$)

$$(F(x_0 + t_n u) - y_0) \bigcap (-\mathrm{int}C) = \emptyset,$$

$$(F(x_0 + t_n u) - y_0) \bigcap (A_F(x_0, y_0)(t_n u) + r_n B_Y) \neq \emptyset.$$

Therefore, there is $P_n \in A_F(x_0, y_0)$ such that, for such n,

$$t_n P_n + o(t_n) \notin -\mathrm{int}C. \tag{3.1}$$

We have two possibilities, for both of which the assumed asymptotic p-compactness is applied as follows. If $\{P_n\}$ is norm bounded we can assume that $P_n \xrightarrow{p} P \in \text{p} - \mathrm{cl}A^S_F(x_0, y_0)$. Passing (3.1) to limit we obtain $P(u) \notin -\mathrm{int}C$ as required. While if $\{P_n\}$ is norm unbounded, we have (by extracting a subsequence if necessary) $P_n/\|P_n\| \xrightarrow{p} P \in \text{p}{-}A^S_F(x_0, y_0)_\infty \setminus \{0\}$. Dividing (3.1) by $\|P_n\|$ and letting $n \to \infty$, we also obtain $P(u) \notin -\mathrm{int}C$.

(ii) Since $(A_F(x_0, y_0), B_F(x_0, y_0))$ is a second-order approximation of F at the local weak solution (x_0, y_0), for a sequence $t_n \to 0^+$ and sufficiently large n one gets (for a positive r_n such that $r_n t_n^{-2} \to 0$)

$$(F(x_0 + t_n u) - y_0) \bigcap (-\text{int}C) = \emptyset,$$

$$(F(x_0 + t_n u) - y_0) \bigcap (t_n A_F(x_0, y_0)(u) + t_n^2 B_F(x_0, y_0)(u, u) + r_n B_Y) \neq \emptyset.$$

Therefore, one has $P_n \in A_F(x_0, y_0)$ and $M_n \in B_F(x_0, y_0)$ such that

$$t_n P_n(u) + t_n^2 M_n(u, u) + o(t_n^2) \notin -\text{int}C.$$

As $A_F(x_0, y_0)(u) \subseteq -C$ one has

$$t_n^2 M_n(u, u) + o(t_n^2) \in -t_n P_n(u) + Y \setminus (-\text{int}C) \subseteq C + Y \setminus (-\text{int}C) \subseteq Y \setminus (-\text{int}C).$$

Now by a usual asymptotic p-compactness argument one gets some $M \in \text{p}-B_F(x_0, y_0)$ such that $M(u, u) \notin -\text{int}C$. □

The following Fritz John optimality condition is necessary for weak efficiency and hence also for all kinds of efficiency mentioned in Definition 3.1.

Theorem 3.4. (Fritz John Necessary Condition). *Impose assumption* (CA). *If* (x_0, y_0) *is a local weakly efficient solution of problem* (P) *and* $z_0 \in G(x_0) \cap -D$*, then* $\forall u \in X$, $\exists P \in \text{p}-A_F(x_0, y_0)$, $\exists Q \in A_G(x_0, z_0)$, $\exists (\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ *such that*

$$\langle \varphi, P(u) \rangle + \langle \psi, Q(u) \rangle \geq 0,$$

$$\langle \psi, z_0 \rangle = 0.$$

Furthermore, for u satisfying $0 \in \text{int}(Q(u) + z_0 + D)$ *for all* $Q \in A_G(x_0, z_0)$*, we have* $\varphi \neq 0_{Y^*}$.

Proof. If $u \in W_f(A, x_0)$, by Proposition 3.3(i), there exists $P \in \text{p}-A_F(x_0, y_0)$ such that $P(u) \notin -\text{int}C$. While if $u \notin W_f(A, x_0)$, by Proposition 3.2(i), there is $Q \in A_G(y_0, z_0)$ with $Q(u) \notin -\text{int}(D + z_0)$. So in both cases, one has some $P \in \text{p}-A_F(x_0, y_0)$ and $Q \in A_G(x_0, z_0)$ such that

$$(P(u), Q(u)) \notin -\text{int}(C \times (D + z_0)).$$

According to the separation theorem we have some $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that

$$\langle \varphi, P(u) \rangle + \langle \psi, Q(u) \rangle \geq 0,$$

$$\langle \psi, z_0 \rangle = 0.$$

Now let u satisfy $0 \in \text{int}(Q(u) + z_0 + D)$ for all $Q \in A_G(x_0, z_0)(u)$ and suppose to the contrary that $\varphi = 0_{Y^*}$. Then the separation result collapses to

$$\langle \psi, Q(u) \rangle \geq 0,$$

$$\langle \psi, z_0 \rangle = 0.$$

This implies that $\psi(Q(u) + z_0 + d) \geq 0$ for all $d \in D$. So $0 \notin \text{int}(Q(u) + z_0 + D)$, which contradicts the assumption. □

Remark 3.5. (i) Since any approximation is contained in the corresponding strong approximation we can state the corresponding necessary condition using strong approximations, but it is weaker than Theorem 3.4.

(ii) Applied to the special case where F is single-valued, Theorem 3.4 improves Theorem 3.1 of [21], since the norm boundedness of $A_G(x_0, z_0)$ required in [21] is omitted.

The following examples supply simple cases where Theorem 3.4 is applicable while several earlier results are not.

Example 3.6. Let $X = Y = Z = \mathbb{R}$, $C = D = \mathbb{R}_+$, $F : X \to 2^Y$ and $G : X \to 2^Z$ be defined by

$$F(x) = \begin{cases} \{y \in \mathbb{R} : y \geq x^2 - 1\}, & \text{if} \quad x \geq 0, \\ \{y \in \mathbb{R} : 2\sqrt[3]{x} + x - 1 \leq y \leq x^2\}, & \text{if} \quad x < 0, \end{cases}$$

$$G(x) = \begin{cases} \{y \in \mathbb{R} : -x - 1 \leq y \leq 0\}, & \text{if} \quad x \geq 0, \\ \{y \in \mathbb{R} : y \geq -x\}, & \text{if} \quad x < 0 \end{cases}$$

and $(x_0, y_0) = (0, -1)$ and $z_0 = -1$. First, we apply the Theorem 3.4. We can take the following approximations of F and G at (x_0, y_0): $A_F(x_0, y_0) = \{1\}$ and $A_G(x_0, z_0) = (-\infty, -1]$. Then, the compactness assumption in Theorem 3.4 is satisfied. Furthermore, we have p-$A_F(x_0, y_0) = \{1\}$. Let us check the assumption of Theorem 3.4 for $u = -1$. We see that, for all $P \in$ p$-A_F(x_0, y_0)$, all $Q \in A_G(x_0, z_0)$ and all $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ with $< \psi, z_0 >= 0$ (i.e. $\psi = 0$),

$$< \varphi, P(u) > + < \psi, Q(u) >= -\varphi < 0.$$

Hence (x_0, y_0) is not a local weak efficient solution.

Trying with other results observe that $A = [0, +\infty)$, $W_f(A, x_0) = T(A, x_0) = [0, +\infty)$. So, for all $P \in$ p$-A_F(x_0, y_0)$ and all $v \in T(A, x_0)$, $P(v) \notin -\text{int}C$. Hence Theorem 3.1 in [22] cannot be applied. Furthermore, we can check that Theorem 4.1 in [22] does not work either.

Now, for all $u \in T(A, x_0) = W_f(A, x_0)$, we compute the Dini derivative $F'(x_0, y_0, u)$ and the extended Dini derivative $D_D(x_0, y_0)(u)$ (introduced in [9]) as follows:

$$F^{'}(x_0, y_0, u) = D_D(x_0, y_0)(u) \subseteq \mathbb{R}_+.$$

They do not meet -int$C$. Hence the necessary conditions in [8] and [9] are satisfied and then (x_0, y_0) is not rejected.

To apply [2] using K-approximating multifunctions (introduced in [2]) we need the condition

$$\text{cone}(X - \{x_0\}) \times \{0_{Y \times Z}\} \bigcap T(\text{epi}(F, G), (x_0, y_0, z_0) = \{(0, 0, 0)\}.$$

This is easily seen violated in the problem of this example. Hence Theorem 15 of [2] is out of use either.

Example 3.7. Let $X = Y = Z = \mathbb{R}$, $C = D = \mathbb{R}_+$, $F : X \to 2^Y$ and $G : X \to 2^Z$ be defined by

$$F(x) = \begin{cases} \{y \in \mathbb{R} : y \geq x - e^x\}, & \text{if} \quad x \geq 0, \\ \{y \in \mathbb{R} : y \leq \sqrt[3]{x} + 1\}, & \text{if} \quad x < 0, \end{cases}$$

$$G(x) = \begin{cases} \{y \in \mathbb{R} : 0 \le y \le e^x\}, & \text{if} \quad x \ge 0, \\ \emptyset, & \text{if} \quad x < 0 \end{cases}$$

and $(x_0, y_0, z_0) = (0, 0, 0)$. We easily calculate p-$A_F(x_0, y_0) = (0, +\infty)$ and $A_G(x_0, z_0) = \{1\}$. Now, taking $u = -1$ we see that $0 \in \text{int}(Q(u) + z_0 + D)$ for all $Q \in A_G(x_0, z_0)$. Hence, we need to check the necessary condition given in Theorem 3.4 only for $\varphi \ne 0_{Y^*}$. For $u = 1$ and for all $P \in$ p$-A_F(x_0, y_0)$, all $Q \in A_G(x_0, z_0)$ and all $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ we have

$$< \varphi, P(u) > + < \psi, Q(u) >= -\varphi P - \psi < 0.$$

According to our Theorem 3.4, we conclude that $(x_0, y_0) = (0, 0)$ is not a local weak efficient solution of problem (P).

Now, we easily see that the variational set of $(F, G)_+$ at (x_0, y_0, z_0) is

$$V^1((F, G)_+, x_0, (y_0, z_0)) = \mathbb{R} \times \mathbb{R}_+.$$

Consequently, Theorem 3.1 of [20] cannot be used to reject (x_0, y_0).

It is also easy to compute the image of X through the contingent derivative of (F, G) at $(x_0, (y_0, z_0))$:

$$D_C((F, G), (x_0, (y_0, z_0)))X = \mathbb{R} \times \mathbb{R}_+.$$

Then, for $(y^*, z^*) = (0, 1) \ne (0_{Y^*}, 0_{Z^*})$ and all $(u, v) \in D_C((F, G), (x_0, (y_0, z_0)))X$, one has $< y^*, u > + < z^*, v >= v \ge 0$ and $< z^*, z_0 >= 0$. This shows that Theorem 3.13 of [10] cannot be employed.

To apply [23] using the weak Clarke epiderivative we need the condition that $G(0) := \{y \in \mathbb{R} : (0, y) \in T(\text{epi}F, (x_0, y_0))\}$ is pointed, which is seen violated in this example. Hence Theorem 5.2 of [23] is not applicable either.

Theorem 3.8. (Second-Order Necessary Condition). *If* (CA) *is fulfilled,* (x_0, y_0) *is a local weak efficient solution of problem* (P) *and* $z_0 \in G(x_0) \cap -D$*, then, for all* $u \in X$ *with* $A_F(x_0, y_0)(u) \subseteq -C$ *and* $A_G(x_0, z_0)(u) \subseteq -(D + z_0)$*, there are* $M \in$ p$-B_F(x_0, y_0)$*,* $N \in B_G(x_0, z_0)$ *and* $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ *such that*

$$\langle \varphi, M(u, u) \rangle + \langle \psi, N(u, u) \rangle \ge 0,$$

$$< \psi, z_0 >= 0.$$

Proof. Let $u \in X$ satisfy the assumption. If $u \in W_f(A, x_0)$, by Proposition 3.3(ii) one can find $M \in$ p$-B_F(x_0, y_0)$ such that $M(u, u) \notin -\text{int}C$. While for weakly infeasible u, since $A_G(x_0, z_0)(u) \subseteq -(D + z_0)$, Proposition 3.2(ii) yields some $N \in B_G(y_0, z_0)$ satisfying $N(u, u) \notin -\text{int}(D + z_0)$. Thus, in both cases, there are $M \in$ p$-B_F(x_0, y_0)$ and $N \in B_G(x_0, z_0)$ such that

$$(M(u, u), N(u, u)) \notin -\text{int}(C \times (D + z_0)).$$

The separation theorem now gives some $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that the conclusion holds.

□

Example 3.9. Let $X = \mathbb{R}^2$, $Y = Z = \mathbb{R}$, $C = D = \mathbb{R}_+$ and $F : X \to 2^Y$, $G : X \to 2^Z$ be defined by

$$F(x_1, x_2) = \begin{cases} \{y \in \mathbb{R} : y \geq -|x_1|^{\frac{3}{2}} + x_2\}, & \text{if} \quad (x_1, x_2) \neq (0, 0), \\ \{0\}, & \text{if} \quad (x_1, x_2) = (0, 0), \end{cases}$$

$$G(x_1, x_2) = [x_1^3 - x_2^2, +\infty)$$

and $x^0 = (0, 0)$, $y_0 = 0$ and $z_0 = 0$. We can take p-$A_F(x^0, y_0) = \{(0 \quad 1)\}$ and $A_G(x^0, y_0) = \{(0 \quad 0)\}$. Then, for $(\varphi, \psi) = (0, 1) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*}\}$, we have $< \psi, z_0 >= 0$ and

$$< \varphi, P(u) > + < \psi, Q(u) >= 0$$

for all $u \in X$, all $P \in$ p$-A_F(x_0, y_0)$ and all $Q \in A_G(x_0, z_0)$. So the first-order condition provided by Theorem 3.4 cannot be in use. To apply Theorem 3.8, as a result of direct computations we have

$$\text{p}-B_F(x^0, y_0) = \left\{ \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} : \beta < 0 \right\} \quad \text{and} \quad B_G(x^0, y_0) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Then, for $u = (1, -1)$, $A_F(x_0, y_0)(u) \subseteq -C$ and $A_G(x_0, z_0) \subseteq -(D + z_0)$. We see that

$$< \varphi, M(u, u) > + < \psi, N(u, u) >= \varphi\beta - \psi < 0.$$

for all $M \in$ p$-B_F(x_0, y_0)$, all $N \in B_G(x_0, z_0)$ and all $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$. According to Theorem 3.8, we conclude that (x^0, y_0) is not a local weak efficient solution of problem (P).

4. Sufficient optimality conditions

In this section, sufficient optimality conditions for local Henig-proper, local strong Henig-proper and local Benson-proper efficiencies, and local firm efficiency are established by using strong approximations. In order to develop sufficient optimality conditions certain relaxed convexity assumptions are usually to be imposed on objective mappings and constraints. For example, in [19, 20] the C-star-shapedness and pseudoconvexity are used along with variational sets. In [6] generalized tangent epiderivatives together with cone convexity are appealed to and in [23], dealing with the weak Clarke epiderivative, cone semilocal convexlikeness assumptions are imposed . Here we make use of C-arcwise-connectedness and pseudoconvexity. We will need the properties of C-arcwise-connected mappings given in the following two propositions.

Proposition 4.1. *Let assumption* (CA) *be satisfied and $F : X \to 2^Y$ be C-arcwise-connected at x_0 on S, where $S \subseteq X$ is arcwise connected at x_0. Assume that $y_0 \in F(x_0)$, that for all $x \in S$ (for $H_{x_0,x}(.)$ associated with the arcwise connectedness) the derivative $H'_{x_0,x}(0^+)$ exists. Assume further that, for all $x \in S$ and all $P \in$ p$-A^S_F(x_0, y_0)_\infty \setminus \{0\}$,*

$$P(H'_{x_0,x}(0^+)) \notin -C.$$

Then, for all $x \in S$,

$$F(x) - y_0 \subseteq \text{p} - \text{cl}A^S_F(x_0, y_0)(H'_{x_0,x}(0^+)) + C.$$

Proof. Let $x \in S$ be fixed. Since $F : X \to 2^Y$ is C-arcwise-connected at x_0 on S, for $t \in [0,1]$ one has

$$(1-t)F(x_0) + tF(x) \subseteq F(H_{x_0,x}(t)) + C.$$

This implies that, for all $y \in F(x)$,

$$y - y_0 \in \frac{F(H_{x_0,x}(t)) - y_0}{t} + C.$$

As $H_{x_0,x}(t) \to x_0$ and $A_F^S(x_0,y_0)$ is an approximation of F at (x_0,y_0), for some positive r with $t^{-1}r \to 0^+$ as $t \to 0^+$,

$$F(H_{x_0,x}(t)) - y_0 \subseteq A_F^S(x_0,y_0)(H_{x_0,x}(t) - H_{x_0,x}(0)) + rB_Y.$$

Then

$$y - y_0 \in A_F^S(x_0,y_0)(\frac{F(H_{x_0,x}(t)) - H_{x_0,x}(0)}{t}) + rt^{-1}B_Y + C.$$

We have the following two cases, to which we apply the usual asymptotic p-compactness argument. If $A_F^S(x_0,y_0)$ is norm unbounded, taking a sequence $t_n \to 0^+$ we have sequences $P_n \in A_F^S(x_0,y_0)$ and $y_n \in B_Y$ such that

$$-y + y_0 + P_n(\frac{F(H_{x_0,x}(t_n)) - H_{x_0,x}(0)}{t_n}) + r_n t_n^{-1} y_n \in -C, \tag{4.1}$$

$$P_n/\|P_n\| \xrightarrow{p} P \in \text{p}{-}A_F^S(x_0,y_0)_\infty \setminus \{0\}.$$

Dividing (4.1) by $\|P_n\|$ and letting $t_n \to 0^+$, we obtain $P(H'_{x_0,x}(0^+)) \in -C$. Since this contradicts the assumption, this first case does not happen. So $A_F^S(x_0,y_0)$ must be norm bounded. Then we also have (4.1) and passing this relation to limit we obtain

$$y - y_0 \in \text{p} - \text{cl}A_F^S(x_0,y_0)(H'_{x_0,x}(0^+)) + C.$$

as required. □

Proposition 4.2. *Let $F : X \to 2^Y$,* (CA) *be satisfied for F $x_0 \in S \subseteq \text{dom}F$ and $y_0 \in F(x_0)$. Assume that, for all $x \in S$ and all $P \in \text{p}{-}A_F^S(x_0,y_0)_\infty \setminus \{0\}$, $P(x - x_0) \notin -C$. Impose further either of the following conditions*

(a) S is star − shaped at x_0 on S and F is C − star − shaped at (x_0,y_0) on S;

(b) F is pseudoconvex at (x_0,y_0). Then, for all $x \in S$,

$$F(x) - y_0 \subseteq \text{p} - \text{cl}A_F^S(x_0,y_0)(x - x_0) + C.$$

Proof. (a) As star-shapedness implies arcwise connectedness, applying Proposition 4.1 with $H_{x_0,x}(t) = (1-t)x_0 + tx$ we arrive at the results.

(b) By the pseudoconvexity, for $(x,y) \in \text{gr}F$ one has $(x,y) - (x_0,y_0) \in T_{\text{epi}F}(x_0,y_0)$, i.e. there are sequences $t_n \to 0^+$ and $(x_n,y_n) \to (x - x_0, y - y_0)$ such that, for all n,

$$y_0 + t_n y_n \in F(x_0 + t_n x_n) + C.$$

Since $x_0 + t_n x_n \to x_0$, for sufficiently large n one has

$$t_n y_n \in F(x_0 + t_n x_n) - y_0 + C \subseteq t_n A^S_F(x_0, y_0)(x_n) + C$$

and hence

$$y_n \in A^S_F(x_0, y_0)(x_n) + C.$$

Applying the asymptotic p-compactness (using the assumption $P(x - x_0) \notin -C$ for all the mentioned x and P to reject the unbounded case) we complete the proof.

□

Theorem 4.3. (Sufficient Condition for Local Henig-Proper Efficiency). *Let* (CA) *be satisfied, $x_0 \in A$, $y_0 \in F(x_0)$ and $z_0 \in F(x_0) \cap -D$. Assume that there exists a pointed convex cone $H \subseteq Y$ with $C \setminus \{0\} \subseteq \mathrm{int}H$. Then (x_0, y_0) is a local Henig-properly efficient solution (relative to H) of problem* (P) *if one of the following conditions (a) and (b) holds.*

(a) $(F, G) : X \to 2^Y$ is $(C \times D)$ − arcwise − connected at x_0; for all $x \in A$, all $(P, Q) \in$ p$-A^S_{(F,G)}(x_0, y_0, z_0)_\infty \setminus \{0\}$ one has $(P, Q)(H'_{x_0,x}(0^+)) \notin -(C \times D)$ and also some $(\varphi, \psi) \in H^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that

$$< \varphi, y > + < \psi, z >> 0 \quad \textit{and} \quad < \psi, z_0 >= 0 \tag{4.2}$$

for all $(y, z) \in \mathrm{p} - \mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(H'_{x_0,x}(0^+))$.

(b) (F, G) is pseudoconvex at $(x_0, (y_0, z_0))$; for all $x \in A$, all $(P, Q) \in$ p$-A^S_{(F,G)}(x_0, y_0, z_0)_\infty \setminus \{0\}$ one has $(P, Q)(x - x_0) \notin -(C \times D)$ and some $(\varphi, \psi) \in H^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that (4.2) *holds for all $(y, z) \in \mathrm{p} - \mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(x - x_0)$.*

Proof. Due to a similarity, we check only part (a). Suppose to the contrary there exist $x_n \xrightarrow{A} x_0$, $y_n \in F(x_n)$ and $z_n \in F(x_n) \cap -D$ such that $y_n - y_0 \in -H \setminus \{0\}$ for all n. Making use of Proposition 4.1, one has

$$(y_n, z_n) - (y_0, z_0) \in \mathrm{p} - \mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(H'_{x_0,x_n}(0^+)) + H \times D.$$

Therefore, there exist $(y'_n, z'_n) \in \mathrm{p} - \mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(H'_{x_0,x_n}(0^+))$ and $(h_n, d_n) \in H \times D$ such that

$$(y_n, z_n) - (y_0, z_0) = (y'_n, z'_n) + (h_n, d_n).$$

Consequently, for all $(\varphi, \psi) \in H^* \times D^*$ and all n,

$$\langle \varphi, y'_n + h_n > + < \psi, z'_n + d_n >=< \varphi, y_n - y_0 > + < \psi, z_n - z_0 >\leq - < \psi, z_0 > .$$

This implies that

$$< \varphi, y'_n > + < \psi, z'_n >\leq - < \varphi, h_n > - < \psi, d_n > - < \psi, z_0 >\leq - < \psi, z_0 >,$$

which contradicts the assumption and the proof is finished.

□

Theorem 4.4. (Sufficient Condition for Local Strong Henig-Proper Efficiency). *Assume* (CA). *Let* $x_0 \in A$, $y_0 \in F(x_0)$ *and* $z_0 \in F(x_0) \cap -D$. *Impose further that* C *has a base* B, $\delta = \inf\{\|b\| : b \in B\}$ *and* $\epsilon \in (0, \delta)$. *Then* (x_0, y_0) *is a local strong Henig-properly efficient solution of problem* (P) *if either of the following conditions holds.*

(a) (F, G) *is* $(C{\times}D)-$*arcwise*$-$*connected at* x_0*; for all* $x \in A$, *all* $(P, Q) \in \mathrm{p}{-}A^S_{(F,G)}(x_0, y_0, z_0)_\infty \setminus \{0\}$, *one has* $(P, Q)(H'_{x_0,x}(0^+)) \notin -(C \times D)$ *and as well some* $(\varphi, \psi) \in C_\epsilon(B)^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ *such that*

$$< \varphi, y > + < \psi, z >\geq 0 \quad \textit{and} \quad < \psi, z_0 >= 0$$

for all $(y, z) \in \mathrm{p} - \mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(H'_{x_0,x}(0^+))$, *where* $H_{x_0,x}$ *is provided by the arc-wise connectedness.*

(b) (F, G) *is pseudoconvex at* $(x_0, (y_0, z_0))$*; for all* $x \in A$, *all* $(P, Q) \in \mathrm{p}{-}A^S_{(F,G)}(x_0, y_0, z_0)_\infty \setminus \{0\}$, *one has* $(P, Q)(x - x_0) \notin -(C \times D)$ *and some* $(\varphi, \psi) \in C_\epsilon(B)^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ *such that*

$$< \varphi, y > + < \psi, z >\geq 0 \quad \textit{and} \quad < \psi, z >= 0.$$

for all $(y, z) \in \mathrm{p} - \mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(x - x_0)$.

Proof. Because of a similarity, the proof of part (b) is omitted. For (a) suppose ad absurdum that there exist sequences $x_n \xrightarrow{A} x_0$, $y_n \in F(x_n)$ and $z_n \in F(x_n) \cap -D$ such that $y_n - y_0 \in -\mathrm{int}C_\epsilon(B)$. By Proposition 4.1, one has

$$(y_n, z_n) - (y_0, z_0) \in \mathrm{p} - \mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(H'_{x_0,x_n}(0^+)) + C \times D.$$

Similarly as for Theorem 4.3, one has $(y'_n, z'_n) \in \mathrm{p} - \mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(H'_{x_0,x_n}(0^+))$ and $(c_n, d_n) \in C_\epsilon(B) \times D$ such that

$$(y_n, z_n) - (y_0, z_0) = (y'_n, z'_n) + (c_n, d_n).$$

Then, for all $(\varphi, \psi) \in C_\epsilon(B)^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ and all n,

$$< \varphi, y'_n + c_n > + < \psi, z'_n + d_n >=< \varphi, y_n - y_0 > + < \psi, z_n - z_0 >< - < \psi, z_0 > .$$

Hence,

$$< \varphi, y'_n > + < \psi, z'_n >< - < \psi, z_0 > .$$

With this contradiction we are done. □

The next example illustrates advantages of Theorem 4.4.

Example 4.5. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $D = \mathbb{R}$, $F : X \to 2^Y$ and $G : X \to 2^Z$ be defined by

$$F(x) = \{(y_1, y_2) \in \mathbb{R}^2 : x \leq y_1 \leq x + x^2, y_2 = x\},$$

$$G(x) = \begin{cases} \{0\}, & \text{if} \quad x \geq 0, \\ \{x^2\}, & \text{if} \quad x < 0 \end{cases}$$

and $x_0 = 0$,$y^0 = (0, 0)$, $z_0 = 0$. Take $B = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 = 1, y_1 \geq 0, y_2 \geq 0\}$ for a base of C and $\epsilon = \dfrac{1}{\sqrt{2}} \in (0, \delta)$, where $\delta = \inf\{\|b\| : b \in B\} = 1$. Then $C_\epsilon(B) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 \geq 0\}$. We can choose, for $\alpha > 0$ rather small and fixed, the strong approximation

$$A^S_{(F,G)}(x_0, y^0, z_0) = \{(y_1, y_2) \in \mathbb{R}^2 : 1 - \alpha \leq y_1 \leq 1 + \alpha, y_2 = 1\} \times \{0\}.$$

Then $A^S_{(F,G)}(x_0, y^0, z_0)_\infty = 0$ and

$$\mathrm{p}-\mathrm{cl}A^S_{(F,G)}(x_0, y^0, z_0) = \{(y_1, y_2) \in \mathbb{R}^2 : 1-\alpha \le y_1 \le 1+\alpha, y_2 = 1\} \times \{0\}.$$

We see that $A = [0, +\infty)$ is star-shaped and (F, G) is $(C \times D)$-star-shaped at x_0 and hence the assumption about arcwise connectedness in Theorem 4.4 is fulfilled. For all $(P, Q) \in \mathrm{p}-A^S_{(F,G)}(x_0, y_0, z_0)_\infty \setminus \{0\}$, one sees that $(P, Q)(H'_{x_0,x}(0^+)) \notin -(C \times D)$. The remaining condition of Theorem 4.4 is satisfied with $(\varphi, \psi) = ((1, 1), 0)$. According to this theorem (x_0, y^0) is a local strong Henig-properly efficient solution of (P).

By direct calculations we see that $(-1, -1, 0) \in V^1((F, G)_+, x_0, (y_0, z_0))$. Then

$$V^1((F, G)_+, x_0, (y_0, z_0)) \cap -(\mathrm{int}C_\epsilon(B) \times D(z_0)) \neq \emptyset.$$

Therefore, Theorem 3.6 of [19] cannot be applied.

Theorem 4.6. (Sufficient Condition for Benson-Proper Efficiency). *Let* $(F, G) : X \to 2^Y \times 2^Z$, $x_0 \in A$, $y_0 \in F(x_0)$ *and* $z_0 \in F(x_0) \cap -D$. *Then* (x_0, y_0) *is a Benson-properly efficient solution of problem* (P) *if either of the following conditions holds.*

(i) (F, G) *is* $(C{\times}D)$*–arcwise–connected at* x_0*; for all* $x \in A$, *all* $(P, Q) \in \mathrm{p}-A^S_{(F,G)}(x_0, y_0, z_0)_\infty \setminus \{0\}$, *one has* $(P, Q)(H'_{x_0,x}(0^+)) \notin -(C \times D)$ *and there exists* $(\varphi, \psi) \in C^{*i} \times D^*$ *such that*

$$< \varphi, y > + < \psi, z > \geq 0 \quad and \quad < \psi, z_0 >= 0$$

for all $(y, z) \in \mathrm{p}-\mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(H'_{x_0,x}(0^+))$, *where* $H_{x_0,x}$ *is provided by the arc-wise connectedness.*

(ii) (F, G) *is pseudoconvex at* $(x_0, (y_0, z_0))$*; for all* $x \in A$, *all* $(P, Q) \in \mathrm{p}-A^S_{(F,G)}(x_0, y_0, z_0)_\infty \setminus \{0\}$, *one has* $(P, Q)(x - x_0) \notin -(C \times D)$ *and there is* $(\varphi, \psi) \in C^{*i} \times D^*$ *such that*

$$< \varphi, y > + < \psi, z > \geq 0 \quad and \quad < \psi, z_0 >= 0$$

for all $(y, z) \in \mathrm{p}-\mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(x - x_0)$.

Proof. Because of a similarity, the proof of part (b) is omitted. For (a) suppose ad absurdum that there exist a nonzero point $y \in \mathrm{clcone}(F(A) + C - y_0) \cap (-C)$. Then there exist positive $\lambda_n, x_n \in A, y_n \in F(x_n), z_n \in F(x_n) \cap (-D)$ and $c_n \in C$ such that

$$\lim_{n\to\infty}\lambda_n(y_n + c_n - y_0) = y. \tag{4.3}$$

By Proposition 4.1, one has

$$(y_n, z_n) - (y_0, z_0) \in \mathrm{p}-\mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(H'_{x_0,x_n}(0^+)) + C \times D.$$

Therefore, there exist $(y'_n, z'_n) \in \mathrm{p}-\mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(H'_{x_0,x_n}(0^+)$ and $(c'_n, d'_n) \in C \times D$ such that

$$(y_n, z_n) - (y_0, z_0) = (y'_n, z'_n) + (c'_n, d'_n).$$

Then, for all $(\varphi, \psi) \in C^{*i} \times D^*$ and all n,

$$< \varphi, y'_n + c'_n > + < \psi, z'_n + d'_n >=< \varphi, y_n - y_0 > + < \psi, z_n - z_0 > .$$

Furthermore, from (4.3) one has

$$\lim_{n\to\infty}\lambda_n(<\varphi, y_n - y_0> + <\varphi, c_n>) =<\varphi, y><0.$$

Therefore, for sufficiently large n, $<\varphi, y_n - y_0><0$. This implies that

$$<\varphi, y_n^{'} + c_n^{'}> + <\psi, z_n^{'} + d_n^{'}> =<\varphi, y_n - y_0> + <\psi, z_n - z_0>< - <\psi, z_0> .$$

and hence

$$<\varphi, y_n^{'}> + <\psi, z_n^{'}>< - <\psi, z_0> .$$

With this contradiction we arrive at the result. □

Example 4.7. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $D = \mathbb{R}_+$, $F : X \to 2^Y$ and $G : X \to 2^Z$ be defined by $G(x) = \{-x\}$,

$$F(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq x^2, y_2 = -x\}, & \text{if} \quad x > 0, \\ \{(0,0)\}, & \text{if} \quad x \leq 0, \end{cases}$$

and $x_0 = 0$, $y^0 = (0,0)$, $z_0 = 0$. We can choose, for $\alpha > 0$ small and fixed, the strong approximation

$$A^S_{(F,G)}(x_0, y^0, z_0) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 > \alpha, y_2 = -1\} \times \{-1\}.$$

Then

$$\mathrm{p} - \mathrm{cl}A^S_{(F,G)}(x_0, y^0, z_0) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq \alpha, y_2 = -1\} \times \{-1\},$$

$$\mathrm{p}{-}A^S_{(F,G)}(x_0, y^0, z_0)_\infty = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0, y_2 = 0\} \times \{0\}.$$

We see that $A = [0, +\infty)$ is star-shaped and (F, G) is $(C \times D)$-star-shaped at x_0 and hence the assumption about arcwise connectedness in Theorem 4.6 is fulfilled. For all $(P, Q) \in \mathrm{p}{-}A^S_{(F,G)}(x_0, y_0, z_0)_\infty \setminus \{0\}$, one sees that $(P, Q)(H^{'}_{x_0,x}(0^+)) \notin -(C \times D)$. The remaining condition of Theorem 4.6 is satisfied with $(\varphi, \psi) = ((\alpha + \alpha^{-1}, 1), 0) \in C^{*i} \times D^*$, since, for all $((y_1, y_2), z) \in \mathrm{p} - \mathrm{cl}A^S_{(F,G)}(x_0, y_0, z_0)(x - x_0)$,

$$<\varphi, (y_1, y_2)> + <\psi, z> = y_1(\alpha + \alpha^{-1}) - 1 > \alpha(\alpha + \alpha^{-1}) - 1 = \alpha^2 > 0.$$

According to this theorem (x_0, y^0) is a Benson-properly efficient solution of (P). By direct calculations of variational sets and contingent derivatives we see that $(0, -1) \in V^1(F, x_0, y_0), -1 \in V^1(G, x_0, z_0)$ and $(0, -1) \in D_C F(x_0, y_0)(1), -1 \in D_C G(x_0, z_0)(1)$. Then, for all $c^* \in C^{*i}$ and $d^* \in D^*$,

$$\inf[c^* V^1(F, x_0, y_0) + d^* V^1(G, x_0, z_0)] < 0,$$

$$\inf_{x \in A}[c^* D_C F(x_0, y_0)(x) + d^* D_C G(x_0, z_0)] < 0.$$

Therefore, Theorem 4.3 of [3] and Theorem 2 of [25] cannot be in use.

Now we pass to local firm efficiency. In the theorem below, we don't impose explicitly any convexity assumption. However, we need X to be finite dimensional. First we characterize contingent cone $T(A, x_0)$ in terms of approximations of G (instead of derivatives of G as usual).

Proposition 4.8. *Let $x_0 \in A$ and $z_0 \in G(x_0) \cap -D$. Then, for all $u \in T(A, x_0)$, there exists $Q \in \mathrm{p}{-}A^S_G(x_0, z_0)$ such that $Q(u) \in -D(z_0)$.*

Proof. If $u \in T(A, x_0)$, then there are $t_n \to 0^+$ and $u_n \to u$ such that, for all n, $x_0 + t_n u_n \in A$, i.e. $G(x_0 + t_n u_n) \cap -D \neq \emptyset$. For sufficiently large n and a positive r_n with $r_n t_n^{-1} \to 0$, one sees that

$$G(x_0 + t_n u_n) - z_0 \subseteq t_n A_G^S(x_0, z_0)(u_n) + r_n B_Z.$$

Therefore, one has $Q_n \in A_G^S(x_0, z_0)$ and $z_n \in r_n B_Z$ such that

$$z_0 + t_n Q_n(u_n) + z_n \in -D$$

and hence

$$Q_n(u_n) + z_n/t_n \in -D(z_0).$$

By the asymptotic p-compactness one gets some $Q \in \text{p}-A_G^S(x_0, z_0)$ such that $Q(u) \in -D(z_0)$ as desired.

□

Theorem 4.9. (Sufficient Condition for Local Firm Efficiency of Order 1). *Assume that X is finite dimensional, $x_0 \in A$, $y_0 \in \mathrm{StrMin}_C F(x_0)$ and $z_0 \in G(x_0) \cap -D$. Assume further that, for all $u \in T(A, x_0)$ with norm one and for all $(P, Q) \in \text{p}-A_{(F,G)}^S(x_0, y_0, z_0)$, there exists $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that*

$$< \varphi, P(u) > + < \psi, Q(u) >> 0 \quad \textit{and} \quad < \psi, z_0 >= 0.$$

Then, (x_0, y_0) is a local firm efficient solution of order 1 of problem (P).

Proof. Arguing by contraposition suppose there exist sequences $x_n \in A$ tending to x_0, $c_n \in C$ and $y_n \in F(x_n)$ such that

$$y_n - y_0 + c_n \in B_Y(0, n^{-1} t_n),$$

where $t_n = \|x_n - x_0\|$. Since X is finite dimensional, $u_n := (x_n - x_0)t_n^{-1}$ converges to some $u \in T(A, x_0)$ of norm one. By the definition of $A_F^S(x_0, y_0)$ one has a positive r_n with $r_n t_n^{-1} \to 0$ such that

$$y_n - y_0 \in A_F^S(x_0, y_0)(x_n - x_0) + r_n B_Y.$$

So there exist $P_n \in A_F^S(x_0, y_0)$ and $\overline{y_n} \in r_n B_Y$ such that

$$t_n P_n u_n + \overline{y_n} + c_n \in B_Y(0, n^{-1} t_n).$$

Hence

$$P_n u_n + \overline{y_n}/t_n \in -C + B_Y(0, n^{-1}).$$

By the asymptotic p-compactness one gets some $P \in \text{p}-A_F^S(x_0, y_0)$ such that $P(u) \in -C$. As $u \in T(A, x_0)$, in view of Proposition 4.8 there exists $Q \in \text{p}-A_G^S(x_0, z_0)$ such that $Q(u) \in -D(z_0)$. Hence, $(P(u), Q(u)) \in -(C \times D(z_0))$. So for any $(\varphi, \psi) \in C^* \times D^*$ with $< \psi, x_0 >= 0$ one has

$$< \varphi, P(u) > + < \psi, Q(u) > \leq 0,$$

which is impossible.

□

Corollary 4.10. *Assume that X is finite dimensional, $x_0 \in A$, $y_0 \in F(x_0)$ and $z_0 \in G(x_0) \cap -D$. Assume further that C is pointed, F is C-calm at (x_0, y_0) and, for all $u \in T(A, x_0)$ with norm one and all $(P, Q) \in$ p$-A^S_{(F,G)}(x_0, y_0, z_0)$, there exists $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that*

$$< \varphi, P(u) > + < \psi, Q(u) >> 0 \quad \textit{and} \quad < \psi, z_0 >= 0.$$

Then, (x_0, y_0) is a local firm efficient solution of order 1 of problem (P).

Proof. To apply Theorem 4.9 we need to prove only that $y_0 \in \mathrm{StrMin}_C F(x_0)$. Suppose there exists $y \in F(x_0)$ such that $y - y_0 \in -C \setminus \{0\}$. As F is C-calm at (x_0, y_0), $y - y_0 \in C$. So $y - y_0 \in C \cap (-C \setminus \{0\})$, which contradicts the pointedness of C.

□

Now we compare Theorem 4.9 with recent existing results by an example.

Example 4.11. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $D = \mathbb{R}_+$, $F : X \to 2^Y$ and $G : X \to 2^Z$ be defined by $G(x) = \{-x\}$,

$$F(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq x\sqrt{|x|}, y_2 = x^2\}, & \text{if} \quad x > 0, \\ \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = x^3, y_2 = x^2\}, & \text{if} \quad x \leq 0 \end{cases}$$

and $x_0 = 0$, $y^0 = (0, 0) = \mathrm{StrMin}_C F(x_0)$ and $z_0 = 0$. We have the following approximation

$$\text{p}-A^S_{F,G}(x_0, y^0, z_0) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 > 0, y_2 = 0\} \times \{-1\}.$$

As $T(A, x_0) = [0, +\infty)$, only $u = 1$ needs to be considered. Taking $(\varphi, \psi) = ((1, 0), 0) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})$ one has $< \psi, z_0 >= 0$ and

$$< \varphi, P(u) > + < \psi, Q(u) >= y_1 > 0$$

for all $(P, Q) \in$ p$-A^S_{(F,G)}(x_0, y_0, z_0)$. Thus, Theorem 4.9 ensures that $(x_0, y^0) = (0, (0, 0))$ is a local firm efficient solution of order 1 of problem (P).

Trying to apply [10] we compute contingent derivative $D_C(F, G)(x_0, (y^0, z_0))(X \setminus \{0\}) = \mathbb{R}_+ \times \{0\} \times \mathbb{R} \setminus \{0\}$. For $(0, 0, -1) \in D_C(F, G)(x_0, (y^0, z_0))(X \setminus \{0\})$ and all $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ we have $< \psi, z_0 >= 0$ and

$$< \varphi, (0, 0) > + < \psi, -1 >= -\psi \leq 0.$$

So Theorem 3.13 of [10] is out of use. Now taking [11] into account, we need the contingent derivatives of F and F_+ at (x_0, y^0) :

$$D_C F(x_0, y^0)(u) = \mathbb{R}_+ \times \{0\} \quad \text{and} \quad D_C(F + C)(x_0, y^0)(u) = \mathbb{R}_+ \times \mathbb{R}_+.$$

In consequence,

$$(0, 0) \in D_C(F + C)(x_0, y^0)(u) \quad \text{and} \quad D_C(F + C)(x_0, y^0)(u) \cap -C = \{0\}.$$

So both Theorems 5.5 and 5.9 of [11] do not work.

Next, trying with [7] we can see that $(0, 0, -1) \in (F, G)'(x_0, y^0, z_0; 1)$. Consequently,

$$(F, G)'(x_0, y^0, z_0; 1) \cap -(C \times D(z_0)) \neq \emptyset$$

and hence Theorem 4.2 of [7] cannot be employed.

Remark 4.12. We note that the gap between the necessary condition in Theorem 3.4 and the sufficient one in Theorem 4.9 is rather minimal: the inequalities in Theorem 3.4 are replaced by the strict ones in Theorem 4.9 and the $\exists$ by the $\forall$.

Theorem 4.13. (Sufficient Condition for Local Firm Efficiency of Order 2). *Assume that X is finite dimensional, $x_0 \in A$, $y_0 \in \mathrm{StrMin}_C F(x_0)$, $z_0 \in G(x_0) \cap -D$. Assume further that, for each $u \in T(A, x_0)$ with norm satisfying $P(u) \in -C$ for some $P \in A^S_F(x_0, y_0)$, the following two conditions hold*

(i) *there exists a neighborhood U of u such that, for all $v \in U$ with $\|v\| = 1$, $A^S_{(F,G)}(x_0, y_0, z_0)(v) \subseteq C \times D(z_0)$;*

(ii) *for all $(M, N) \in$ p$-B^S_{(F,G)}(x_0, y_0, z_0)$, there exists $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ such that $< \psi, z_0 >= 0$ and*

$$< \varphi, M(u, u) > + < \psi N(u, u) >> 0.$$

Then, (x_0, y_0) is a local firm efficient solution of order 2 of (P).

Proof. Suppose there exist sequences $x_n \in A$ satisfying $x_n \to x_0$, $c_n \in C$ and $y_n \in F(x_n)$ such that

$$y_n - y_0 + c_n \in B_Y(0, n^{-1}t_n^2),$$

where $t_n = \|x_n - x_0\|$. We can assume that $u_n := \dfrac{x_n - x_0}{t_n}$ converges to some $u \in T(A, x_0)$ with norm one. Similarly as for Theorem 4.9 there exists $P \in$ p$-A^S_F(x_0, y_0)$ with $P(u) \in -C$. As $(A^S_F(x_0, y_0), B^S_F(x_0, y_0))$ is a second-order strong approximation of F at (x_0, y_0) one has positive r_n such that $r_n t_n^{-2} \to 0$ and

$$y_n - y_0 \in A^S_F(x_0, y_0)(x_n - x_0) + B^S_F(x_0, y_0)(x_n - x_0, x_n - x_0) + r_n B_Y.$$

Therefore, $P_n \in A^S_F(x_0, y_0)$, $M_n \in B^S_F(x_0, y_0)$ and $y'_n \in r_n B_Y$ exist such that

$$t_n P_n(u_n) + t_n^2 M_n(u_n, u_n) + y'_n + c_n \in B_Y(0, n^{-1}t_n^2).$$

By assumption (i) $P_n(u_n) \in C$. Let $c'_n = t_n P_n(u_n) + c_n \in C$. We have

$$M_n(u_n, u_n) + y'_n/t_n^2 + c'_n/t_n^2 \in B_Y(0, n^{-1}).$$

Using the asymptotic p-compactness one gets some $M \in$ p$-B^S_F(x_0, y_0)$ such that $M(u, u) \in -C$. On the other hand, for some positive s_n with $s_n t_n^{-2} \to 0$,

$$G(x_0 + t_n u_n) - z_0 \subseteq A^S_G(x_0, z_0)(x_n - x_0) + B^S_G(x_0, z_0)(x_n - x_0, x_n - x_0) + s_n B_Z.$$

Therefore, there are $Q_n \in A^S_G(x_0, z_0)$, $N_n \in B^S_G(x_0, z_0)$ and $z_n \in s_n B_Z$ such that

$$z_0 + t_n Q_n(u_n) + t_n^2 N_n(u_n, u_n) + z_n \in -D.$$

Since $t_n Q_n(u_n) \in D(z_0)$ (by (i)), this implies that

$$N_n(u_n, u_n) + z_n/t_n^2 \in -D(z_0).$$

Again by the asymptotic p-compactness, one obtains some $N \in$ p$-B^S_G(x_0, z_0)$ such that $N(u, u) \in -D(z_0)$. Therefore, there are $(M, N) \in$ p$-B^S_{(F,G)}(x_0, y_0, z_0)$ such that $(M(u, u), N(u, u)) \in -(C \times D(z_0))$. Hence, for all $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})$, we have $< \psi, z_0 >= 0$ and

$$< \varphi, M(u, u) > + < \psi, N(u, u) >\leq 0,$$

contradicting (ii) and we are done.

□

Example 4.14. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $D = \mathbb{R}_+$, $F : X \to 2^Y$ and $G : X \to 2^Z$ be defined by

$$F(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = x^2, 0 \leq y_2 \leq x^2 + x\}, & \text{if} \quad x \geq 0, \\ \{(x^2, x)\}, & \text{if} \quad x < 0, \end{cases}$$

$$G(x) = \begin{cases} \{0\}, & \text{if} \quad x \geq 0, \\ \{x^2\}, & \text{if} \quad x < 0 \end{cases}$$

and $x_0 = 0$, $y^0 = (0, 0) = \mathrm{StrMin}_C F(x_0)$ and $z_0 = 0$. For α positive and fixed we have a first strong approximation and some related sets for (F, G) as follows

$$A^S_F(x_0, y^0) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = 0, y_2 \in [0, \alpha + 1]\},$$

$$A^S_{(F,G)}(x_0, y^0, z_0) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = 0, y_2 \in [0, \alpha + 1]\} \times \{0\},$$

$$\text{p}-A^S_F(x_0, y^0) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = 0, y_2 \in [0, \alpha + 1]\},$$

$$\text{p}-A^S_{(F,G)}(x_0, y^0, z_0) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = 0, y_2 \in [0, \alpha + 1]\} \times \{0\}.$$

Since $T(A, x_0) = [0, +\infty)$, we need to consider $u = 1 \in T(A, x_0)$. Theorem 4.9 is out of use, as for all $(\varphi, \psi) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})\}$ and for $(P, Q) = ((0, 0), 0) \in p - A^S_{(F,G)}(x_0, y^0, z_0)$, one has $< \varphi, Pu > + < \psi, Qu >= 0$.

Now we apply Theorem 4.13. We choose $P = (0, 0) \in p - A^S_{F,G}(x_0, y^0)$ to obtain $P(u) = (0, 0) \in -C$. For any neighborhood U of $u = 1$, one sees that $v = 1 \in U$ satisfies $A^S_{(F,G)}(x_0, y_0, z_0)(v) \subseteq C \times D(z_0)$. Now we can take p-$B^S_{(F,G)}(x_0, y^0, z_0) = \{(1, 0)\} \times \{0, 1\}$ and $(\varphi, \psi) = ((1, 0), 0) \in C^* \times D^* \setminus \{(0_{Y^*}, 0_{Z^*})$. Then, for all $(M, N) \in$ p$-B^S_{(F,G)}(x_0, y^0, z_0)$,

$$< \varphi, M(u, u) > + < \psi, N(u, u) >= 1 > 0 \quad \textit{and} \quad < \psi, z_0 >= 0.$$

Thus, Theorem 4.13 ensures that $(x_0, y^0) = (0, (0, 0))$ is a local firm efficient solution of order 2 of problem (P).

Trying to apply [10] we can see that F and G are calm at (x_0, y^0) and (x_0, y_0), respectively. But we have $(0, -1, 0) \in D^2_C(x_0, y^0, z_0)(0, 0, 0)(-1)$ and so

$$D^2_C(x_0, y^0, z_0)(0, 0, 0)(-1) \cap ((-C) \times (-D - \mathbb{R}_+ z_0)) \neq \emptyset.$$

Therefore Theorem 3.12 of [10] says nothing about (x_0, y^0).

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